A Simulation of the 't Hooft Model at finite N_c with the Overlap Dirac Operator*

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We present some results of a numerical investigation of the 't Hooft model with 2, 3 and 4 colors, regularized on the lattice with overlap fermions.

Several years ago, in a pioneering investigation, 't Hooft studied $U(N_c)$ theories in the limit $N_c \to \infty$ with $C_t = g^2 N_c$ kept constant [1]. In two dimensions this led him to a model of QCD that exhibited important features of the full theory. Recently, the overlap formulation has made it possible to introduce a lattice Dirac operator D which in the massless limit preserves a lattice form of chiral symmetry at finite lattice spacing [2]. This development, together with a renewed interest in 't Hooft's results, prompted us to perform a numerical investigation of a class of two-dimensional non-Abelian models with overlap lattice femions. Precisely, we simulated models of QCD_2 with $SU(N_c)$ and $U(N_c)$ color groups for $N_c = 2, 3, 4$, in the 't Hooft limit $C_t = g^2 N_c = \text{const}$. Since in two dimensions only the $U(N_c)$ models exhibit a $U(1)$ axial anomaly, we studied both classes of models in order to analyze the behavior of the overlap operator in presence and absence of the axial anomaly. Our systems are small enough that it is possible to compute exactly in the overlap formulation the propagator D^{-1} and $\det(D)$, following the scheme used in [3]. We computed the masses of the pions as a function of the quark mass m and, for both classes of models, we found the expected functional dependence on m and universality in N_c. In the $U(N_c)$ case the meson spectrum exhibits also a flavor singlet particle (η'), whose mass is known analytically, and which we computed by following the approximate method of Ref. [4] and by using the Witten-Veneziano relation. The details of our simulation, as well as additional results that we could not include here because of lack of space, will be presented in a separate article [5].

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1. The 't Hooft Model with Overlap Fermions

The 't Hooft model action in the overlap regularization reads

$$
\begin{aligned}
S &= \beta \sum_{x,\mu<\nu} \left[1 - \frac{1}{2N_c} \text{Re Tr } U_{\mu\nu}(x)\right] \qquad (1)\\
&+ \sum_{x,y} \overline{\psi}(x) \left[(1 - \frac{ma}{2}) D_{xy} + m\ \delta_{xy}\right] \psi(y)
\end{aligned}
$$

where $U_{\mu\nu}$ is the usual Wilson plaquette, a is the lattice spacing and ψ are N_f fermion fields with mass m and $\beta = 2N_c/(ag)^2$. $D = (1+V)/a$ is the massless Neuberger-Dirac operator, V being the unitary component of $D_W - 1/a$, where D_W is the usual Wilson-Dirac operator.

For $m = 0$ the action (2) is invariant under a continuous symmetry $\delta\psi = \gamma_5(1 - aD)\psi, \quad \delta\overline{\psi} = \overline{\psi}\gamma_5$, *i.e.* a form of chiral symmetry that holds at finite lattice spacing. The $N_f = 2$ models exhibit an isotriplet of particles (π), whose masses vary with the quark mass as [1] [6]

$$
M_\pi^2 = 2\sqrt{\frac{C_t\ \pi}{3}}\ m + \dots\ , \quad U(N_c) \text{ models} \qquad (2)
$$

$$
M_\pi^2 = \frac{9}{\pi}\ (2^7 C_t)^{\frac{1}{3}} (\frac{e^\gamma}{\pi})^{\frac{4}{3}}\ m^{\frac{4}{3}} + \dots\ , \quad SU(N_c) \text{ models} \qquad (3)
$$

Eq.(2) has been derived for $N_c \to \infty$ while Eq.(3) also involves a semiclassical WKB approximation.

In the $U(N_c)$ models in the chiral limit $M_{\eta'}^2 = N_f\ g^2/\pi$. The Witten-Veneziano formula for the

$U(N_c)$ models gives

$$M^2_{\eta'} = \frac{4N_f}{f^2_\pi} \frac{\langle Q^2 \rangle}{V} \tag{4}$$

2. The meson spectrum

We performed extensive simulations with the exact Neuberger-Dirac operator on a 18×18 lattice at $C_t = 4/3$, *i.e.* $\beta = 6,\ 13.5,\ 24$ for $N_c = 2, 3, 4$ respectively. We computed M_π for $m = 0.04, 0.05, 0.06, 0.07, 0.08, 0.1$ so that the relation $N_x M_\pi > 4$ holds. The effects of dynamical fermions have been included by weighting the observables with the appropriate power of the fermion determinant. The smallness of the lattice warrants this procedure (cfr. [3]), which would lead to unacceptable variance on larger systems.

We computed the meson propagators projected over zero momentum from the fermion propagators and extracted the meson masses in a standard manner. The errors have been estimated with the jacknife analysis.

In table (1) we report the pion masses for the $U(N_c)$ and $SU(N_c)$ models with two flavors of dynamical fermions and in the quenched approximation. The quenched results get closer to the unquenched two flavors results when N_c gets larger, as one would expect since the large N_c limit corresponds to discarding the internal quark loops. For the $U(N_c)$ case Eq.(2) motivated us to do the fit

$$\frac{M^2_\pi}{g^2 N_c} = A + B\ \frac{m}{g\sqrt{N_c}} \tag{5}$$

We obtained $A = -0.012(17),\ B = 1.57(19)$ for $N_c = 2$, $A = -0.006(4),\ B = 1.53(4)$ for $N_c = 3$ and $A = -0.006(8),\ B = 1.66(8)$ for $N_c = 4$.

For the $SU(N_c)$ models, following Eq.(3), we performed the fit

$$\frac{M_\pi}{g\sqrt{N_c}} = A + B\ (\frac{m}{g\sqrt{N_c}})^{\frac{2}{3}} \tag{6}$$

We obtained $A = 0.022(26),\ B = 1.524(120)$ for $N_c = 2$, $A = 0.045(43),\ B = 1.548(166)$ for $N_c = 3$ and $A = -0.009(40),\ B = 1.788(11)$ for $N_c = 4$. In the $SU(N_c)$ case we performed also the fit $M_\pi/g\sqrt{N_c} = C(m/g\sqrt{N_c})^\gamma$ and we obtained γ compatible with $2/3$ for $N_c = 2, 3, 4$. Figures(1,2) give evidence of universality in N_c for the pion masses.

$U(N_c)$ models			
$m_q/g\sqrt{N_c}$	$N_c = 2$ M^2_π/g^2N_c	$N_c = 3$ M^2_π/g^2N_c	$N_c = 4$ M^2_π/g^2N_c
$N_f = 2$			
0.0346	0.045(10)	0.047(2)	0.055(6)
0.0433	0.056(9)	0.059(2)	0.067(5)
0.0519	0.068(8)	0.072(3)	0.079(4)
0.0606	0.080(7)	0.085(3)	0.093(3)
0.0692	0.094(5)	0.098(3)	0.107(2)
0.0866	0.124(4)	0.128(3)	0.138(1)
quenched ($N_f = 0$)			
0.0346	0.069(1)	0.063(1)	0.057(1)
0.0433	0.083(1)	0.076(1)	0.071(1)
0.0519	0.098(1)	0.091(1)	0.085(1)
0.0606	0.113(1)	0.106(1)	0.100(1)
0.0692	0.128(1)	0.121(1)	0.116(1)
0.0866	0.161(1)	0.153(1)	0.148(1)
$SU(N_c)$ models			
$(m_q/gN_c^{\frac{1}{2}})^{2/3}$	$N_c = 2$ $M_\pi/gN_c^{\frac{1}{2}}$	$N_c = 3$ $M_\pi/gN_c^{\frac{1}{2}}$	$N_c = 4$ $M_\pi/gN_c^{\frac{1}{2}}$
$N_f = 2$			
0.1062	0.183(14)	0.206(23)	0.181(28)
0.1233	0.211(12)	0.237(23)	0.211(26)
0.1392	0.236(10)	0.263(21)	0.239(24)
0.1543	0.258(8)	0.286(19)	0.266(22)
0.1686	0.279(7)	0.307(16)	0.292(21)
0.1957	0.321(5)	0.348(10)	0.341(17)
quenched ($N_f = 0$)			
0.1062	0.200(2)	0.212(1)	0.217(1)
0.1233	0.227(2)	0.241(1)	0.246(1)
0.1392	0.252(2)	0.267(1)	0.273(1)
0.1543	0.276(1)	0.292(1)	0.298(1)
0.1686	0.299(1)	0.316(1)	0.322(1)
0.1957	0.343(1)	0.361(1)	0.367(1)

Table 1

In the $U(N_c)$ models we computed also $M_{\eta'}$. The η' propagator is given by the difference between the connected and disconnected terms in the correlator and, because of cancellations, the errors are larger than in the π case. Therefore, we resorted the method proposed in [4] to compute the η' mass. The method exploits the quenched two-loop disconnected $\Gamma_q^{2-\mathrm{loop}}$ and

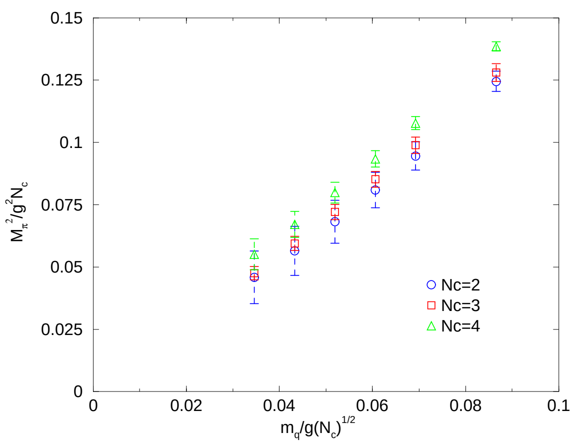

Figure 1. M_π^2/g^2N_c *vs.* $m_q/g\sqrt{N_c}$ *for* $N_f = 2$, $U(N_c)$ *models*

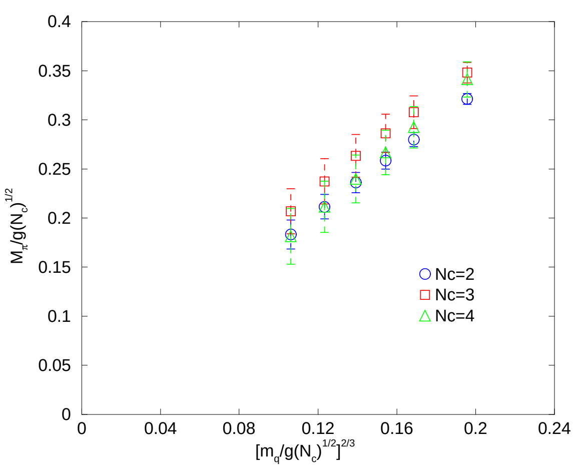

Figure 2. $M_\pi/g\sqrt{N_c}$ *vs.* $(m_q/g\sqrt{N_c})^{2/3}$ *for* $N_f = 2$, $SU(N_c)$ *models*

$N_c = 2$ $M_{\eta'}^2/g^2N_c$	$N_c = 3$ $M_{\eta'}^2/g^2N_c$	$N_c = 4$ $M_{\eta'}^2/g^2N_c$
Analytic		
0.159	0.106	0.079
Numerical from Eq.(7)		
0.159(8)	0.111(7)	0.084(8)
Numerical from Eq.(4)		
0.129(6)	0.097(7)	0.076(6)

Table 2

quenched one-loop connected $\Gamma_q^{1-\text{loop}}$ contributions to the η' propagator:

$$M_{\eta'}^2 = 2M_\pi \lim_{t\to\infty} \frac{\Gamma_q^{2-\text{loop}}(t)}{|t|\Gamma_q^{1-\text{loop}}(t)} \tag{7}$$

As one can see from table (2) $M_{\eta'}^2$ computed using Eq.(7) and $N_f = 1$ compares remarkably well with the analytic result.

An alternative calculation of $M_{\eta'}$ can be done by using the Witten-Veneziano formula. In the quenched approximation we computed $f_\pi = 0.774(3),\ 0.956(3),\ 1.115(5)$ for $N_c = 2, 3, 4$ respectively. Since in our calculations we diagonalize the full Neuberger-Dirac operator, we could determine the number of zero modes, and thus the topological charge, for each individual gauge configuration. We thus obtained $\langle Q^2\rangle/V = 0.0258(16),\ 0.0298(23),\ 0.0319(37)$ for $N_c = 2, 3, 4$ respectively. (Notice that with increasing N_c these values get closer and closer to the analytic value $\langle Q^2\rangle/V = C_t/4\pi^2 = 0.0337$. Building on the trade-off between spatial and internal degrees of freedom that prompted the introduction of the $N_c \to \infty$ single plaquette model [7], one could heuristically argue that the theory should be less affected by finite size effects increasing N_c. A rigorous justification of this argument, however, would require a detailed analysis of discretization and finite size effects that goes beyond the scope of our investigation.) From f_π and $\langle Q^2\rangle/V$, we use Eq.(4) to calculate $M_{\eta'}$. The results, also in table (2), are consistent for $N_c = 3$ and 4 with those obtained with the method of Ref. [4] and with the analytical formula.

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